Some Facts About the Mandelbrot Set

Mitch Richling

0.1 About This Document?

This short note is intended to document several facts regarding the Mandelbrot set that might be of interest to someone wishing to draw the set efficiently with a computer. I make no attempt to justify or develop any results here; this is simply a list of results. I have attempted to use the notation most commonly found in books on the subject.

0.2 What is the Mandelbrot Set?

Before we can define the Mandelbrot set, we need to define some notation for function iteration. Let f be a function. Now we can define

$$\begin{aligned} f^{(1)}(z) &= f(z) \qquad \text{and} \\ f^{(n)}(z) &= f(f^{(n-1)}(z)) \end{aligned}$$

For example, $f^{(2)}(z) = f(f(z))$ and $f^{(3)}(z) = f(f(f(z)))$

Several equivalent definitions for the Mandelbrot set exist, but here we use the most common definition. First we define $f : \mathbb{C} \to \mathbb{C}$, for some fixed $c \in \mathbb{C}$, as:

$$f(z) = z^2 + c$$

Now the Mandelbrot set, $M \subset \mathbb{C}$, may be defined to be:

$$M = \left\{ c \in \mathbb{C} : \lim_{n\to\infty} f^{(n)} \text{ is finite} \right\}$$

0.3 Integer Escape Function

Pictures of the Mandelbrot set are often very colorful. What can the colors possibly represent given that the Mandelbrot set is nothing more than a "simple" set in the complex plane? The most important thing to realize is that the colors almost never represent the Mandelbrot set, but rather some artifact of the method used to approximate the set. The most typical source of color is from the function defined in this section, which is closely related to the convergence of the sequence found in the Mandelbrot set definition previously given.

We define $\mathbb{Z}^+ = \mathbb{N} \cup 0$, $\mathbb{R}^+ = (0, \infty]$, and $f(z) = z^2 + c$ as before. For any fixed $b \in \mathbb{R}^+$ we are now ready to define the integer escape function $L_b : \mathbb{C} \to \mathbb{Z}^+$ by

$$L_b(c) = \begin{cases} \min\left(\left\{ n \in \mathbb{Z}^+ : |f^{(n)}| > b \right\}\right) & \text{if } c \notin M \\ 0 & \text{if } c \in M \end{cases}$$

That is to say, L is defined to be the lest integer such that $f^{(n)}$ is outside of the disk of radius b centered at 0.

We recall that $B(0, b) = \{z \in \mathbb{C} : |z| < b\}$, i.e. a ball centered at 0 with radius b. It may be shown that $M \subset B(0, 2)$, and this makes $L(2, c)$ a very special case. It is for this reason that $L(2, c)$ is the source for the color schemes used for most Mandelbrot set images.

0.4 Potential Function

Let $\widehat{L}_b : \mathbb{C} \to \mathbb{R}$ denote the natural extension of L_b to $\mathbb{R}$. Clearly $\widehat{L}_b$ is not continuous, and this explains why the colors found surrounding most Mandelbrot set images are not "smooth". In this section we present a continuous function, G, that is most commonly interpreted as the "electrostatic potential" of the Mandelbrot set. We define the function $G : \mathbb{C} \to \mathbb{R}$ by

$$G(c) = \lim_{n\to\infty} \frac{\log|f^{(n)}|}{2^n}$$

It is easy to show that all level curves of G, also called equipotential curves of M, are topological circles. From this fact it follows that M is connected.

0.5 Escape Time Function

The function presented in this section has the flavor of L but the continuity of G. We define the escape time function by:

$$E(c) = -\log_2(G(c))$$

0.6 Distance Bounds

The function G leads to a distance bound result.

Let $c \in \mathbb{C} \setminus M$, the the distance $d(c, \partial M)$ can be bounded by

$$\frac{\sinh G(c)}{2e^{G(c)}|G'|} \leq d(c, \partial M) \leq \frac{2\sinh G(c)}{|G'|}$$

A similar result exists for $c \in M$, but is a bit more complex.

0.7 Random Results

This section enumerates several random results that may be useful.

1. M is open.
2. M is connected.
3. M is simply connected.
4. M is bounded.
5. M is contained in a disk centered at 0 of radius 2 (i.e. $B(0, 2)$).